TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, December 1969

ANTICHAINS AND PRODUCTS IN PARTIALLY ORDERED SPACES(1)

BY
E. D. TYMCHATYN

1. **Introduction.** By a *partially ordered space* X we mean a Hausdorff space X with a partial order which is closed when regarded as a subset of $X \times X$. In this paper X will denote a compact partially ordered space. The symbol 2^X will denote the space of compact subsets of X with the finite topology [3]. This topology has as a basis all finite tuples $\langle V_1, \ldots, V_n \rangle$ where each V_i is an open subset of X. A closed subset A of X is in $\langle V_1, \ldots, V_n \rangle$ if A meets V_i for each i and if $A \subset V_1 \cup \cdots \cup V_n$. The space 2^X is compact and Hausdorff. If X is metric then 2^X is metric and the finite topology on 2^X is the topology induced by the Hausdorff metric.

For each $x \in X$ we let

$$L(x) = \{y \in X \mid y \leqq x\}, \qquad M(x) = \{y \in X \mid x \leqq y\}$$

and $\Gamma(x) = L(x) \cup M(x)$. If $A \subset X$ we let $L(A) = \bigcup \{L(x) \mid x \in A\}$. We define $M(A)$ and $\Gamma(A)$ similarly. We let $L = \{x \in X \mid L(x) = \{x\}\}$ and we let

$$M = \{x \in X \mid M(x) = \{x\}\}.$$

We call L (resp. M) the set of minimal (resp. maximal) elements of X.

A *chain* is a totally ordered set. An *order arc* is a compact and connected chain. It is easy to check that a nondegenerate and separable order arc is homeomorphic to the unit interval. An *antichain* is a totally unordered set. It is known [5] that each chain is contained in a maximal chain. Each maximal chain is a closed set which meets both L and M. Thus L and M are maximal antichains.

Let X be a compact metric partially ordered space such that for each $x \in X$ $\Gamma(x)$ is connected. We prove that the family of compact maximal antichains of X covers X if and only if L and M are closed. The family of compact maximal antichains of X admits the structure of a topological lattice. If L and M are closed and the maximal chains of X are pairwise disjoint order arcs we prove that X is homeomorphic to $M \times [0, 1]$ (the Cartesian product). We study the role of those partially ordered spaces in which the maximal chains are pairwise disjoint. In §5 we use these results to get a characterization of the 2-cell.

We shall need the following theorem which is due to R. J. Koch. This theorem also appears in [6].

Received by the editors May 6, 1969.

(1) This research was supported in part by a grant from the National Science Foundation.

THEOREM 1.1 (KOCH). *If X is a compact partially ordered space such that L is closed and for each $x \in X$, $L(x)$ is connected, then each maximal order arc in X meets L.*

DEFINITION 1.2. If X is a compact partially ordered space let $\mathscr{M}(X)$ denote the set of those order arcs in X which meet both L and M. We consider $\mathscr{M}(X)$ as a subspace of 2^X.

The next three lemmas appear in [7].

LEMMA 1.3. *If X is a compact partially ordered space then the family of compact chains of X is a compact subset of 2^X.*

LEMMA 1.4. *Let X be a compact partially ordered space. If there exists a family $\mathscr{C}$ of maximal chains of X such that $X \subset \bigcup \mathscr{C}$ and $\mathscr{C}$ is a compact subset of 2^X then L and M are compact.*

LEMMA 1.5. *If X is a compact partially ordered space such that L and M are closed then $\mathscr{M}(X)$ is a compact subset of 2^X.*

2. **The product theorem.** We let $[0, 1]$ denote the unit interval with its usual order.

DEFINITION 2.1. If Y is a compact Hausdorff space we let $P(Y)$ denote the space $Y \times [0, 1]$ with the product topology and with the partial order $(y, a) \leqq (x, b)$ if and only if $y = x$ and $a \leqq b$. Then $P(Y)$ is a compact partially ordered space.

LEMMA 2.2. *If X is a compact partially ordered space such that L is closed and for each $x \in X$, $L(x)$ has a unique minimal element then L is a retract of X.*

Proof. For each $x \in X$ let $\pi(x)$ denote the unique minimal element of $L(x)$. This defines a function $\pi: X \rightarrow L$ such that π leaves every element of L fixed.

We must show that π is continuous. Let $x \in X$ and let x_i be a net converging to x. Let z be a cluster point of the net $\pi(x_i)$. Then $z \in L(x)$ since the partial order on X is closed and $z \in L$ since L is closed. Hence $z = \pi(x)$ and π is continuous.

DEFINITION 2.3. A metric r for a partially ordered space X is *radially convex* if $x \leqq y \leqq z$ implies $r(x, y) + r(y, z) = r(x, z)$.

The basic theorem about radially convex metrics is due to J. H. Carruth [1].

THEOREM 2.4. *Every compact, metric, partially ordered space X admits a radially convex metric.*

LEMMA 2.5. *Let X be a compact partially ordered space which contains a compact maximal antichain C. If T is an order arc which meets both L and M then T meets C.*

Proof. Let $P = \{x \in T \mid x \in L(C)\}$ and let $Q = \{x \in T \mid x \in M(C)\}$. Then P and Q are closed. Since C is a maximal antichain $P \cup Q = T$. Now $T \cap L \subset P$ and $T \cap M \subset Q$. Since T is connected $P \cap Q$ is nonvoid. If $z \in P \cap Q$ then $z \in C$ since C is an antichain.

THEOREM 2.6. *Let X be a compact metric partially ordered space such that L and M are compact subsets of X and for each $x \in X$, $\Gamma(x)$ is connected. Then there exists a continuous one-to-one function f: $[0, 1] \to 2^X$ such that*

(i) *for each $a \in [0, 1]$ $f(a)$ is a compact maximal antichain of X,*
(ii) *if $a < b$ in $[0, 1]$ then $f(a) \subset L(f(b))$,*
(iii) *$f(0) = L$ and $f(1) = M$,*
(iv) *$X \subset \bigcup \{f(a) \mid a \in [0, 1]\}$,*
(v) *if $M \cap L$ is void and if $a < b$ in $[0, 1]$ then $f(a) \cap f(b)$ is void.*

Proof.

Case 1. Suppose $M \cap L$ is void. Let Y be the compact Hausdorff space obtained from X by pinching L and M to points [2, p. 98]. Let π be the projection of X onto Y. Define a partial order on Y by setting $x \leqq y$ in Y if and only if there exist $x' \in \pi^{-1}(x)$ and $y' \in \pi^{-1}(y)$ with $x' \leqq y'$. Then Y is a compact metric partially ordered space with unique minimal element $\pi(L)$ and unique maximal element $\pi(M)$.

By Theorem 2.4, Y admits a radially convex metric r such that $r(\pi(L), \pi(M)) = 1$. For each $a \in [0, 1]$ let $B(a) = \{y \in Y \mid r(\pi(L), y) = a\}$ and let $f(a) = \pi^{-1}(B(a))$.

Since r is a radially convex metric $B(a)$ is an antichain in Y and $f(a)$ is an antichain in X. By Koch's theorem $f(a)$ is a maximal antichain. Since $B(a)$ is compact and π is continuous, $f(a) = \pi^{-1}(B(a))$ is compact. Everything else is clear except that f is continuous.

Let $a_i)$ be a sequence which converges to a in $[0, 1]$. Let $x_i)$ be a sequence which converges to x in X such that for each i, $x_i \in f(a_i)$. Since π is continuous

$$r(\pi(x), \pi(L)) = \lim r(\pi(x_i), \pi(L)) = \lim a_i = a.$$

Hence $\pi(x) \in B(a)$ and $x \in f(a)$. Thus each cluster point in 2^X of the sequence $f(a_i))$ is contained in $f(a)$. Let $x \in f(a)$. By Koch's theorem there exists an order arc T such that $x \in T$ and T meets both L and M. For each i, $f(a_i)$ meets T by Lemma 2.5. For each i let $y_i \in T \cap f(a_i)$ and let y be a cluster point of the sequence $y_i)$. Since T is closed $y \in T$. By the above argument $y \in f(a)$. Since $f(a)$ is an antichain $y = x$. Thus, the sequence $f(a_i))$ converges to $f(a)$ in 2^X and f is continuous.

Case 2. Suppose $M \cap L$ is nonvoid. Make the disjoint union of $P(L)$ and X into a partially ordered space by extending the partial orders on $P(L)$ and X so that if $(n, a) \in P(L)$ and $x \in X$ then $(n, a) \leqq x$ if and only if $n \leqq x$ in X. Form the adjunction space X' of $P(L)$ and X by identifying $(n, 1)$ with n for each $n \in L$. The partial order on $X \cup P(L)$ induces a partial order $\leqq'$ on X' such that $(X', \leqq')$ satisfies all the hypotheses of the theorem. The set of minimal (resp. maximal) elements of X' is $L \times \{0\}$ (resp. M). We may apply Case 1 to the partially ordered space X' to obtain a continuous one-to-one function f': $[0, 1] \to 2^{X'}$ such that conditions (i)–(v) are satisfied. For each $a \in [0, 1]$ let

$$f''(a) = \{x \in f'(a) \cup L \mid x \text{ is maximal in } f'(a) \cup L\}.$$

Then $f''(a)$ is a compact maximal antichain. Using f'' it is easy to define a continuous one-to-one function f: $[0, 1] \to 2^X$ such that conditions (i)–(iv) are satisfied.

COROLLARY 2.7. *Let X be a compact metric partially ordered space such that L and M are closed subsets of X and for each $x \in X$, $\Gamma(x)$ is a nondegenerate order arc. Then X is homeomorphic under an order preserving map to $P(M)$.*

Proof. Let f: $[0, 1] \to 2^X$ be a one-to-one continuous function such that conditions (i)–(v) of Theorem 2.6 are satisfied. Define $h: P(M) \to X$ by letting $h(m, a) \in L(m) \cap f(a)$ for each $(m, a) \in P(M) = M \times [0, 1]$.

Let $(m, a) \in M \times [0, 1]$ and let $(m_i, a_i))$ be a sequence converging to (m, a). Let z be a cluster point of $h(m_i, a_i))$. The sequence $m_i)$ converges to m in X. For each i $h(m_i, a_i) \in L(m_i)$. Since the partial order on X is closed $z \in L(m)$. Since the sequence $f(a_i)$ converges to $f(a)$ in 2^X, $z \in f(a)$. Thus $z = h(m, a)$ and f is continuous. By conditions (iv), (ii) and (v) of Theorem 2.6 h is onto X, one-to-one and order preserving. Since $P(M)$ is compact and X is Hausdorff, h is a homeomorphism.

The hypotheses of Corollary 2.7 may be relaxed as follows:

THEOREM 2.8. *Let X be a compact metric partially ordered space such that for each $x \in X$, $\Gamma(x)$ is a chain. If X contains a compact maximal antichain A then X may be embedded by an order preserving map into $P(A)$.*

Proof. By Theorem 2.4 there exists a radially convex metric r for X such that the diameter of X with respect to r is less than $1/2$.

Define g: $X \to A$ by letting $g(x) \in \Gamma(x) \cap A$ for each $x \in X$. By Lemma 2.2 g is a retraction of X onto A.

Define f: $X \to P(A)$ as follows: For $x \in X$ let

$$f(x) = (g(x), 1/2 + r(x, g(x))) \quad \text{if } g(x) \leqq x \quad \text{and}$$
$$f(x) = (g(x), 1/2 - r(x, g(x))) \quad \text{if } x \leqq g(x).$$

Since g is continuous and r is a metric for X, f is continuous. Because r is radially convex f is one-to-one and order preserving. Since X is compact and $P(A)$ is Hausdorff, f is open.

The following theorem is the main result in this paper.

THEOREM 2.9. *Let X be a compact metric partially ordered space such that L and M are closed and disjoint and for each $x \in X$, $\Gamma(x)$ is connected. There exists an order preserving map g: $P(\mathscr{M}(X)) \to X$ such that*

(a) *for each $T \in \mathscr{M}(X)$, g maps the maximal chain $\{T\} \times [0, 1]$ of $P(\mathscr{M}(X))$ homeomorphically onto the maximal chain T of X,*

(b) *for each $y \in [0, 1]$, $g(\mathscr{M}(X) \times \{y\})$ is a compact maximal antichain of X, and*

(c) *the function f: $[0, 1] \to 2^X$ defined by letting $f(y) = g(\mathscr{M}(X) \times \{y\})$ satisfies conditions* (i)–(v) *of Theorem* 2.6.

Proof. Let $f: [0, 1] \to 2^X$ be a function which satisfies conditions (i)–(v) of Theorem 2.6. By Lemma 1.5, $\mathcal{M}(X)$ is a compact subset of 2^X. Define

$$g: P(\mathcal{M}(X)) \to X$$

by letting $g(T, a)$ be the unique element in $T \cap f(a)$ for each $(T, a) \in P(\mathcal{M}(X))$.

Let $(T_i, a_i))$ be a net in $P(\mathcal{M}(X))$ which converges to (T, a) and let z be a cluster point of the net $g(T_i, a_i))$. Since the nets $T_i)$ and $f(a_i))$ converge in 2^X to T and $f(a)$ respectively, $z \in T \cap f(a)$ and g is continuous. Everything else is now clear from the definition of g.

We can prove analogous theorems for much more general situations.

Theorem 2.10. *Let X be a compact partially ordered space and let $\mathcal{C}$ be a compact family (in 2^X) of closed chains of X such that $X \subset \bigcup \mathcal{C}$. Let $g: Y \to \mathcal{C}$ be a continuous function from a compact Hausdorff space Y onto $\mathcal{C}$. There exists a compact partially ordered space*

$$Y^* = \{(y, z) \mid y \in Y \text{ and } z \in g(y)\}$$

and an order preserving map f of Y^* onto X such that

(i) *the maximal chains of Y^* are pairwise disjoint,*

(ii) *Y is homeomorphic to both the set of maximal elements of Y^* and to the set of minimal elements of Y^*, and*

(iii) *for each $y \in Y$, f maps the maximal chain $\{(y, z) \mid z \in g(y)\}$ of Y^* homeomorphically onto the chain $g(y)$ in X.*

Theorem 2.10 is useful where $\mathcal{C}$ can be taken to be a subset of $\mathcal{M}(X)$. If $\mathcal{C}$ has a topological property P then one may be able to prove that Y^* and hence X has property P. Local connectedness and connectedness are two such properties. We shall give some more detailed examples in the next section.

Proof of 2.10. Let the set $Y \times X$ have the product topology and the partial order $(a, b) \leqq (c, d)$ if and only if $a = c$ and $b \leqq d$. Then $Y \times X$ is a compact partially ordered space. Let $Y^* = \{(y, z) \in Y \times X \mid z \in g(y)\}$.

Let $(y_i, z_i))$ be a net in Y^* which converges to (y, z). Then the net $y_i)$ converges to y in Y and the net $z_i)$ converges to z in X. Since g is continuous the net $g(y_i))$ converges to $g(y)$ in 2^X. Hence $z \in g(y)$ and $(y, z) \in Y^*$. Thus Y^* with the relative topology and the partial order inherited from $Y \times X$ is a compact partially ordered space.

The maximal chains in Y^* are the sets $\{(y, z) \mid z \in g(y)\}$ where $y \in Y$. The set of maximal elements of Y^* is

$$\{(y, z) \mid y \in Y \text{ and } z \text{ is the maximal element of } g(y)\}.$$

Let $(y_i, z_i))$ be a net of maximal elements of Y^* which converges to (y, z). As above, the net $g(y_i))$ converges to $g(y)$ in 2^X and the net $z_i)$ converges to z in X. For each i, $g(y_i) \subset L(z_i)$. Since the partial order on X is closed $g(y) \subset L(z)$. Since

$z \in g(y)$, z is the maximal element of $g(y)$ in X. Hence, (y, z) is a maximal element of Y^* and the set of maximal elements of Y^* is compact.

Define a function $f\colon Y^* \to X$ by letting $f(y, z)=z$ for each $(y, z) \in Y^*$. Then, f is continuous since f is the restriction to Y^* of the natural projection of $Y \times X$ onto X. The rest is now clear.

3. *L*-**continuity.** The partial order on X is said to be *L-continuous* if the function $f\colon X \to 2^X$ defined by $f(x)=L(x)$ is continuous.

LEMMA 3.1 (WARD [6]). *If the partial order on a compact partially ordered space X is L-continuous then L is closed.*

We prove a partial converse to Lemma 3.1.

PROPOSITION 3.2. *Let X be a compact partially ordered space. If L is closed and for each $x \in X$, $L(x)$ is an order arc, then the partial order on X is L-continuous.*

Proof. Let $x_i)$ be a net which converges to x in X. Since 2^X is compact the net $L(x_i))$ has a cluster point A in 2^X. Since the partial order on X is closed $A \subset L(x)$. By a result in [3] A is connected. Clearly $x \in A$. For each i, $L(x_i) \cap L$ is nonvoid and L is closed. It follows that A meets L. Since $L(x)$ is a continuum which is irreducible with respect to containing x and $L(x) \cap L$, $A=L(x)$. Thus the partial order on X is L-continuous.

THEOREM 3.3. *Let X be a compact partially ordered space such that for each $x \in X$, $L(x)$ is a chain. Then* (i) *is equivalent to* (ii) *and* (iii) *implies* (i):

(i) *the family $\mathscr{C}$ of maximal chains of X is closed in 2^X,*

(ii) *M is closed and the partial order on X is L-continuous,*

(iii) *the family of compact maximal antichains of X covers X.*

Proof. By Lemma 1.4 and Lemma 3.1, L and M are closed if either (i) or (ii) holds.

(i) implies (ii). Let $x_i)$ be a net converging to x in X. For each i, let $m_i \in M \cap M(x_i)$ and let m be a cluster point of $m_i)$. Then $m \in M \cap M(x)$. We may suppose $m_i)$ converges to m. Since $\mathscr{C}$ is compact and the partial order on X is closed $L(m_i)$ converges to $L(m)$. Hence $L(x_i))$ converges to $L(x)$ and the partial order on X is L-continuous.

(ii) implies (i). Define $f\colon M \to \mathscr{C}$ by letting $f(m)=L(m)$ for each $m \in M$. Since the partial order on X is L-continuous, f is a homeomorphism of the compact space M onto $\mathscr{C}$. Thus $\mathscr{C}$ is compact.

(iii) implies (i). Let $C_i)$ be a net of members of $\mathscr{C}$ which converges to some $C \in 2^X$. By Lemma 1.3, C is a chain. Let $m \in M$ such that $C \subset L(m)$ and let $x \in L(m)$. Let A be a compact maximal antichain which contains x. For each i, $M(A) \cap C_i \cap M$ is nonvoid. Since $L(C_i \cap M)=C_i$ is a chain $C_i \cap A$ is nonvoid. For each i let $a_i \in C_i \cap A$. Each cluster point of $a_i)$ is in $A \cap C$. Now $A \cap C \subset L(m) \cap A$.

Since $x \in L(m) \cap A$, A is an antichain and $L(m)$ is a chain it follows that a_i) converges to x. Thus $L(m)=C$ and $C \in \mathscr{C}$. Hence $\mathscr{C}$ is compact.

PROPOSITION 3.4. *Let X be a compact partially ordered space such that L and M are closed and for each $x \in X$, $\Gamma(x)$ is an order arc. Define π: $X \to M$ by letting $\pi(x) \in M \cap M(x)$ for each $x \in X$. Then π is continuous and open.*

Proof. By Lemma 3.2, π is continuous.

Let $x \in X$ and let U be a neighbourhood of x. Then $\langle U, X\rangle$ is a neighbourhood of $\Gamma(x)$ in 2^X. By Theorem 3.2 there exists a neighbourhood V of $\pi(x)$ such that $\Gamma(V) \subset \langle U, X\rangle$. Then $V \cap M$ is a neighbourhood of $\pi(x)$ in M such that $V \cap M \subset \pi(U)$. Thus π is open.

PROPOSITION 3.5. *Let X be a compact partially ordered space such that for each $x \in X$, $\Gamma(x)$ is a real order arc. If L and M are compact and first countable* (*resp. separable*) *then X is first countable* (*resp. separable*).

Proof. Use Theorem 3.2 and a result of Nachbin [4, p. 48].

In Theorem 2.6 we gave sufficient conditions for a metric partially ordered space to be covered by compact maximal antichains. We would like to extend this result to nonmetric partially ordered spaces.

PROBLEM 1. Let X be a compact partially ordered space such that L and M are closed and for each $x \in X$, $\Gamma(x)$ is a nondegenerate order arc. Does each point of X lie in a compact maximal antichain?

PROBLEM 2. Let X be a compact partially ordered space such that L and M are compact metric spaces. Is X metric if, for each $x \in X$, $\Gamma(x)$ is a separable order arc? In view of Theorem 2.10 it suffices to consider the case where M is homeomorphic to Cantor's ternary set.

The answer to Problem 2 is affirmative in case there exists a family $\mathscr{F}$ of compact maximal antichains such that $X \subset \bigcup \mathscr{F}$ and $\mathscr{F}$ is a compact subset of 2^X. L. Mohler and L. E. Ward, Jr. suggested the following example. It shows that the conditions on X can not be relaxed.

EXAMPLE 3.6. Let X be the set $[0, 1] \times [0, 1]$ with the topology of the comb space. A basic open neighbourhood of $(x, 0)$ has the form

$$(U \times [0, 1]) \setminus \bigcup_{i=1}^{n} (\{x_i\} \times [y_i, z_i])$$

where U is a neighbourhood of x in $[0, 1]$, $x_i \in U$, $0 < y_i \leqq z_i \leqq 1$ and n is a nonnegative integer. If $0 < y \leqq 1$ a basic open neighbourhood of (x, y) has the form $\{x\} \times V$ where V is a neighbourhood of y in $[0, 1]$. Give X the partial order $(m, x) \leqq (n, y)$ if and only if $m=n$ and $x \leqq y$. Then X is a compact partially ordered space such that $L=[0, 1] \times \{0\}$ and $M=[0, 1] \times \{1\}$. Notice that X has all the properties of the space in Problem 2 except that M is not closed. However, X is not separable.

4. **Compact maximal antichains.** We shall define lattice operations on the space of compact maximal antichains of a compact partially ordered space X.

DEFINITION 4.1. *A topological lattice* is a Hausdorff space X with two continuous, associative, commutative and idempotent operations $\vee$ and $\wedge$ such that for $x, y \in X$, $x \vee (y \wedge x) = x \wedge (y \vee x) = x$ and the conditions $x \wedge y = y$ and $x \vee y = x$ are equivalent.

DEFINITION 4.2. If X is a compact partially ordered space let $\mathscr{A}(X)$ denote the space of compact maximal antichains of X with its relative topology as a subset of 2^X. For $C, D \in \mathscr{A}(X)$ let

$$C \vee D = \{x \in C \cup D \mid x \text{ is maximal in } C \cup D\}$$

and let

$$C \wedge D = \{x \in C \cup D \mid x \text{ is minimal in } C \cup D\}.$$

LEMMA 4.3. *Let X be a compact partially ordered space. If $C, D \in \mathscr{A}(X)$ then $C \vee D$ and $C \wedge D$ are compact antichains.*

Proof. Clearly $C \vee D$ and $C \wedge D$ are antichains. Let $x_i)$ be a net in $C \vee D$ which converges to $x \in X$. We may suppose that for each i, $x_i \in C$. Since C is closed $x \in C$. For each i, let $y_i \in D \cap L(x_i)$ and let y be a cluster point of $y_i)$. Then $y \in D \cap L(x)$. Since C and D are antichains x is maximal in $C \cup D$. Thus $x \in C \vee D$ and $C \vee D$ is closed. Similarly $C \wedge D$ is closed.

LEMMA 4.4. *Let X be a compact partially ordered space such that L and M are closed and for each $x \in X$, $\Gamma(x)$ is connected. If $C, D \in \mathscr{A}(X)$ then $C \vee D$ and $C \wedge D$ are in $\mathscr{A}(X)$.*

Proof. Let $x \in X$. By Koch's theorem there exists an order arc T such that $x \in T$ and T meets both L and M. By Lemma 2.5 there exists $y \in T \cap C$ and $z \in T \cap D$. Since T is a chain $x \in \Gamma(C \vee D) \cap \Gamma(C \wedge D)$. By Lemma 4.3, $C \wedge D$ and $C \vee D$ are antichains hence $C \wedge D$ and $C \vee D$ are in $\mathscr{A}(X)$.

THEOREM 4.5. *Let X be a compact partially ordered space such that L and M are closed and for each $x \in X$, $\Gamma(x)$ is connected. Then $\mathscr{A}(X)$ with operations $\wedge$ and $\vee$ is a topological lattice with minimal element L and maximal element M. If X is metric then $\mathscr{A}(X)$ is arcwise connected.*

Proof. By Lemma 4.4 $\wedge$ and $\vee$ are operations on $\mathscr{A}(X)$. The proof that $\vee$ and $\wedge$ are associative operations is similar to the proof of Lemma 4.4.

Let $C_i)$ and $D_i)$ be nets in $\mathscr{A}(X)$ which converge to C and D respectively. Let H be a cluster point of the net $C_i \vee D_i)$ in 2^X and let $x \in H$. For each i let $x_i \in C_i \vee D_i$ such that $x_i)$ converges to x. Then $x \in C \cup D$. If $x \in C \cap D$ then $x \in C \vee D$. If $x \in D \backslash C$ then eventually $x_i \in D_i \backslash C_i$. Thus $x_i \in M(C_i)$ eventually. Since $C_i)$ converges to C and the partial order on X is closed $x \in M(C)$. Thus $x \in C \vee D$ and $H \subset C \vee D$. For each i and each maximal order arc T $(C_i \vee D_i) \cap T$ is nonvoid by Lemma 2.5. It follows that $H \cap T$ is nonvoid. Hence H is a maximal antichain. Since $H \subset C \vee D$,

$H=C\vee D$ and $\vee$ is a continuous operation on $\mathscr{A}(X)$. Similarly $\wedge$ is a continuous operation on $\mathscr{A}(X)$.

If X is metric then by Theorem 2.6 there exists an arc in $\mathscr{A}(X)$ which contains the maximal and minimal elements of $\mathscr{A}(X)$. It follows that $\mathscr{A}(X)$ is arcwise connected.

5. A characterization of the 2-cell.

THEOREM 5.1. *Let X be a nondegenerate metric partially ordered space. Suppose there exists a map f: $[0, 1] \to 2^X$ such that*

(i) *for each $a \in [0, 1]$, $f(a) \in \mathscr{M}(X)$,*

(ii) *if $a \leqq b \leqq c$ in $[0, 1]$ then $f(a) \cap f(c) \subset f(b)$ and*

(iii) $X \subset \bigcup f([0, 1])$.

If X has no cutpoints then X is homeomorphic to the closed 2-cell.

Proof. It is well known (see Franklin [9]) that the relation R: $[0, 1] \to X$ defined by letting $R(a)=f(a)$ for each $a \in [0, 1]$ is upper semicontinuous. Since $R(a)$ is compact and connected for each $a \in [0, 1]$, $X=R([0, 1])$ is compact and connected by [9].

If $a \in [0, 1]$ such that $f(0)\neq f(a)\neq f(1)$ then $f(a)$ separates X since $f([0, a]) \cap f([a, 1])=f(a)$. Since X has no cutpoints it follows that if $f(a)$ is a point then either $f(a)=f(0)$ or $f(a)=f(1)$.

The set $\{f(a) \mid a \in [0, 1]\}$ is a compact family of maximal chains of X such that $X \subset \bigcup \{f(a) \mid a \in [0, 1]\}$. By Lemma 1.4, L and M are compact.

Consider the function π: $[0, 1] \to M$ defined by letting $\pi(a) \in f(a) \cap M$ for each $a \in [0, 1]$. Let $a_i)$ be a net in $[0, 1]$ which converges to a. For each i, $f(a_i) \subset L(\pi(a_i))$. If z is a cluster point of $\pi(a_i))$ then $f(a) \subset L(z)$ since the partial order on X is closed. Now $\pi(a) \in f(a)$ so $\pi(a) \in L(z)$. Since $\pi(a) \in M$ it follows that $z=\pi(a)$ and π is continuous. By condition (ii) of the hypotheses π is monotone. Hence, M is either a point or an arc. Similarly, L is either a point or an arc. Thus, $f(0) \cup f(1) \cup M \cup L$ is a simple closed curve.

Suppose $L \cap M$ is empty. By Theorem 2.9 there exists a map g: $P([0, 1]) \to X$ such that for each $a \in [0, 1]$, g maps $\{a\} \times [0, 1]$ homeomorphically onto $f(a)$.

For each $x \in X$, $g^{-1}(x)$ is a compact antichain. Also, for each $x \in X$

$$\{a \in [0, 1] \mid x \in f(a)\}$$

is connected by condition (ii) of the hypotheses. It is now easy to check that g is monotone. By a Theorem of R. L. Moore (Whyburn [8, p. 173]) X is a 2-cell.

If $L \cap M$ is nonvoid then we must go back to the methods used in Case 2 of Theorem 2.6 to obtain a monotone map of $P([0, 1])$ onto X. The argument is completely straightforward.

COROLLARY 5.2. *Let X be a nondegenerate compact metric space with no cutpoints. Then X is a closed 2-cell if and only if X admits a closed partial order such that*

(i) *M is an arc,*
(ii) *L is closed,*
(iii) *for each $m \in M$, $L(m)$ is an order arc,*
(iv) *$L(m)$ separates X for each cutpoint m of M.*

Proof. The partially ordered space $P([0, 1])$ is a partially ordered space on the 2-cell which satisfies all the hypotheses of the theorem.

If X has a partial order satisfying the hypotheses of the theorem let h be a homeomorphism of $[0, 1]$ onto M and define f: $[0, 1] \to 2^X$ by letting $f(a) = L(h(a))$ for each $a \in [0, 1]$. By Theorem 3.2, f is continuous. Condition (iv) of the hypotheses implies that if $a \leqq b \leqq c$ in $[0, 1]$ then $f(a) \cap f(c) \subset f(b)$. Hence Theorem 5.1 applies.

References

1. J. H. Carruth, *A note on partially ordered compacta*, Pacific J. Math. **24** (1968), 229–231. MR **36** #5902.

2. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR **16**, 1136.

3. E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182. MR **13**, 54.

4. L. Nachbin, *Topology and order*, Van Nostrand, Princeton, N. J., 1965. MR **36** #2125.

5. L. E. Ward, Jr., *Partially ordered topological spaces*, Proc. Amer. Math. Soc. **5** (1954), 144–161. MR **16**, 59.

6. ———, *Concerning Koch's theorem on the existence of arcs*, Pacific J. Math. **15** (1965), 347–355. MR **31** #6206.

7. L. E. Ward, Jr. and E. D. Tymchatyn, *On three problems of S. P. Franklin and A. D. Wallace*, Colloq. Math. **20** (1969), 229-236.

8. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I., 1942. MR **4**, 86.

9. S. P. Franklin, *Concerning continuous relations*, Dissertation, University of California, Los Angeles, Calif., 1963.

University of Saskatchewan,
Saskatoon, Canada
University of Oregon,
Eugene, Oregon